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ENTRANCE EXAMINATION, 2012

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) The answer must be written in the space provided in the answer table for Section—A and in the space given after each question of Section—B and of Section—C. Answer written in any other place will not be evaluated.

(iii) For each question in Section—A, *one and only one* of the four choices given is the correct answer. Indicate the correct answer for (a), (b), (c) or (d). Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. If a question is not attempted, then no marks will be awarded for it.

(iv) Questions in Section—B have short answers and each question carries 2 marks.

(v) Answers to all the questions in Section—C must be **justified with mathematical reasoning**, or else they will be considered **invalid**. They carry 4 marks each.

(vi) In the following, $\mathbf{N}$, $\mathbb{Z}$, $\mathbf{Q}$, $\mathbb{R}$ and $\mathbf{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. Subsets of $\mathbf{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(vii) By $[a, b] \subset \mathbb{R}$ we denote the interval of all the real numbers between a and b including the end points. In order to exclude the end points $\{a, b\}$ from the interval, the notation (a, b) is used.

(viii) Extra pages are attached at the end of the question paper for rough work.

SECTION—A

1. Let G be a finite group and let $O(G)$ denote the order of G. Which of the following is true?

   (a) $O(G) = 8 \Rightarrow G$ is non-Abelian

   (b) $O(G) = 21 \Rightarrow G$ is Abelian

   (c) $O(G) = 9 \Rightarrow G$ is Abelian

   (d) $O(G) = 6 \Rightarrow G$ has a unique proper normal subgroup

2. For the closed unit disk D in $\mathbf{R}^2$, which of the following subsets has positive Lebesgue measure?

   (a) S^1, the unit circle

   (b) $D \setminus \{(x, y) \in D \mid x \in \mathbf{R} \setminus \mathbf{Q}\}$

   (c) $D \setminus \{(x, y) \in D \mid x \notin \mathbf{Q}, y \notin \mathbf{Q}\}$

   (d) $D \setminus \{(x, y) \in D \mid x^2 + y^2 \in \mathbf{Q}\}$

3. Let A and B be linear operators on a finite-dimensional vector space V over $\mathbf{R}$ such that $AB = (AB)^2$. If BA is invertible, then which of the following is true?

   (a) $BA = AB$ on V

   (b) $\mathrm{Tr}(A)$ is non-zero

   (c) 0 is an eigenvalue of B

   (d) 1 is an eigenvalue of A

4. The number of elements of order 6 in $\mathbb{Z}/12\mathbb{Z}$ is

   (a) 2

   (b) 3

   (c) 4

   (d) 6

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5. Let $A \in M_{2\times 4}(\mathbb{R})$ and $B \in M_{4\times 2}(\mathbb{R})$. Then the rank of BA is

(a) at least 1

(b) at most 2

(c) exactly 3

(d) at least 4

6. Let X be the set of roots of unity in $\mathbb{C}$. Let $S(X)$ be the set of all sequences of elements in X. Which of the following subsets of $S(X)$ is countable?

(a) The set A of all $(x_n) \in S(X)$ such that (x_n) is an eventually constant sequence

(b) The set B of all $(x_n) \in S(X)$ such that $x_n = 1$ whenever n is a prime number

(c) The set C of all $(x_n) \in S(X)$ such that each x_n is a 26th root of unity

(d) The set D of all $(x_n) \in S(X)$ such that $x_{2n} = 1$ for all $n \geq 1$

7. Let $\mathbb{F}$ be a field with 729 elements. How many distinct proper subfields does $\mathbb{F}$ contain?

(a) 2

(b) 3

(c) 1

(d) 4

8. Two cards are drawn one by one at random from a standard deck of 52 cards. What is the probability that the first card is an ace or the second card is an ace?

(a) $\frac{30}{221}$

(b) $\frac{31}{221}$

(c) $\frac{32}{221}$

(d) $\frac{33}{221}$

9. Let $G = A_4$ be the alternating group of permutations on 4 symbols. Let $n = \max\{o(g) \mid g \in G\}$, where $o(g)$ denotes the order of an element $g \in G$. Then the value of n is

(a) 12

(b) 4

(c) 3

(d) 6

10. Let E be a subset of $\mathbb{R}$. Which of the following statements are equivalent?

(S1) E is connected.

(S2) E is path connected.

(S3) E is closed.

(S4) E is convex.

(S5) E is uncountable.

(a) S3 and S5

(b) S2, S3 and S4

(c) S1, S2 and S4

(d) S4 and S5

11. Let X be a metric space and $E \subset X$. Then which of the following are equivalent?

(S1) E is closed and bounded.

(S2) E is compact.

(S3) E is bounded.

(S4) E has finite intersection property.

(S5) Every infinite subset of E has a limit point in E.

(S6) E is closed.

(a) S1 and S2

(b) S1, S3 and S6

(c) S1 and S5

(d) S2, S4 and S5

12. Let f be a holomorphic (complex analytic) function on $\mathbf{C}$ such that $f(x) = x^2$ for all $x \in \mathbb{R}$ with $x \geq 0$. Then which of the following is/are correct?

(S1) $\operatorname{Re} f(z) = (\operatorname{Re} z)^2$ for all $z \in \mathbf{C}$

(S2) $f(z) = z^2$ on $\mathbf{C}$

(S3) f is real valued

(S4) The Cauchy-Riemann equations of f have a zero of order 2

(a) S3 and S4

(b) S1

(c) S2

(d) None of the above

13. For a commutative ring R, let $R[X]$ denote the polynomial ring in one variable over R. Let I be the ideal $\langle 5, X^2 + 3X - 1 \rangle$ in $\mathbb{Z}[X]$. Then which of the following hold(s) for the ring $\mathbb{Z}[X]/I$?

(S1) It is isomorphic to $(\mathbb{Z}/5\mathbb{Z})[X]$.

(S2) It has 5 zero divisors.

(S3) It is a finite field.

(S4) It has no element of finite order.

(a) S1 and S2

(b) S3

(c) S4

(d) None of the above

14. Let $f : (0, 1) \to [0, 1]$ be a continuous function. Which of the following is/are true?

(S1) f is uniformly continuous.

(S2) f has a fixed point.

(S3) f is a differentiable function.

(S4) There is a continuous function $\tilde{f} : [0, 1] \to [0, 1]$ such that $\tilde{f} = f$ on $(0, 1)$.

(a) S1 and S2

(b) S3

(c) S4

(d) None of the above

15. In the set $M_2(\mathbb{R})$, which of the following sets contain a singular matrix?

$X = \{A \mid A^2 = \mathrm{Id}\}$
$Y = \{A \mid {}^tA = A\}$
$Z = \{A \mid {}^tAA = A{}^tA = \mathrm{Id}\}$
$W = \{A \mid AB = BA, \text{ for all } B \in M_2(\mathbb{R})\}$
$T = \{A \mid A \text{ has at least one eigenvalue which is not real}\}$

(a) X

(b) Y and W

(c) $Z \cap W$

(d) T

16. Let $f : \mathbb{R} \to \mathbb{R}$ is such that f is open. Then which of the following is/are true?

(S1) f is one to one.

(S2) f is onto.

(S3) f has exactly two zeros.

(S4) f is bounded.

(a) S1

(b) S2 and S3

(c) S4

(d) None of the above

SECTION—C

C1. Find all complex polynomials $p \in \mathbb{C}[z]$ such that $p(z_1 + z_2) = p(z_1) + p(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

C2. Let (X, d) be a compact metric space. Let $f : X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto.

C3. Define $f : \mathbf{R} \to \mathbf{R}$ as follows :

$$f(x) = \begin{cases} x^2(\log_e x)^2 & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Find a point x such that $f(x)$ attains a local maximum value. Find the area enclosed between the x-axis and the curve of $f(x)$.

C4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous map such that $f(x) = f(x^3)$ for all $x \in [0, 1]$. Find the value of $f(1) - f(0)$.

SECTION—B

B1. Write the logical opposite (that is, negation) of the following statement :

"If E is a non-empty set, then it is finite."

B2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -12 & 35 \\ 0 & -6 & 17 \end{pmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

B3. Let X be a non-empty set with a relation ~ on it. Suppose ~ is both symmetric and transitive. Prove or disprove that ~ is an equivalence relation.

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ENTRANCE EXAMINATION, 2011

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Mathematical Sciences

[Field of Study Code : MATP (160)]

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INSTRUCTIONS FOR CANDIDATES

(i) This question paper consists of two Parts—Part A and Part B.

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PART—A

Answer **all** questions. Each question carries *6* marks

A1. *(a)* For a metric space (X, d), let $d^2(x, y) := (d(x, y))^2$, for all $x, y \in X$. Is (X, d^2) also a metric space?

(b) Recall that any non-zero $x \in \mathbb{Q}$ can be written as $x = 2^r \cdot (a/b)$, where a and b are both odd integers and $r \in \mathbb{Z}$. Let $\|x\| = 2^{-2r}$, for such an x. Define the function $d_0 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ as

$$d_0(x, y) = \begin{cases} \|x - y\| & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Is $(\mathbb{Q}, d_0)$ a metric space?

A2. Let U be a bounded open disk in $\mathbb{C}$ and let $\mathscr{H}(U)$ denote the ring of complex analytic functions on U.

(a) Is $\mathscr{H}(U)$ an integral domain?

(b) Give an example of a maximal ideal in $\mathscr{H}(U)$.

(c) The ring $\mathscr{H}(U)$ is also a vector space over $\mathbb{C}$. Is it finite dimensional?

A3. Let H be a Hilbert space over $\mathbb{C}$ and let $\mathrm{BL}(H)$ denote the space of bounded linear operators on H. Let $S = \{A \in \mathrm{BL}(H) \mid A^6 = I\}$, where I denotes the identity operator on H. Let T be an operator in S.

(a) Find the spectrum of T.

(b) Is T invertible?

(c) Is S a finite set?

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A4. Let G be a group and let e denote the identity element in G. A subgroup H of G is said to be *characteristic* if any automorphism of G carries H to itself.

(a) Prove that any characteristic subgroup is normal in G.

(b) Prove that the center $Z(G)$ of G is a characteristic subgroup.

(c) Let $S = \{a \in G \mid a^n = e \text{ for some } n \in \mathbf{N}\}$. Prove that the subgroup generated by S is a characteristic subgroup.

A5. Let $f(x) = x^3 - 5 \in \mathbf{Q}[x]$ and let K denote the splitting field of f over $\mathbf{Q}$.

(a) Find $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ (for some $n \in \mathbf{N}$) such that $K = \mathbf{Q}(\alpha_1, \ldots, \alpha_n)$. What is the value of $[K : \mathbf{Q}]$?

(b) Compute $[L : \mathbb{R}]$; L being the splitting field of f over $\mathbb{R}$.

(c) Find $[K(\sqrt{3}) : \mathbf{Q}(\sqrt{3})]$.

PART—B

Answer **all** questions. Each question carries *4* marks

B1. Let $S = \{f : (0, 1) \to \mathbb{R}\}$ be a set of maps. Write the logical opposite (negation) of the following statements :

(a) All maps in S are continuous and at least one map in S is uniformly continuous.

(b) All maps in S are continuous but exactly one map in S is uniformly continuous.

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B2. For $x \in \mathbb{R}$, let $p(x) = x^2 - 2x - 1$ and $f(x) = x^{1/x}$. Find

(i) $\lim_{x \to 0} f(|x|)$

(ii) $\lim_{x \to \infty} p(f(x))$

B3. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$.

(a) Find the set $A = \{P \in \mathbb{R}^2 \mid d(P, Q) \leq 1$ for some $Q \in S\}$.

(b) Is there a continuous map from A onto S?

B4. Let X be a Hausdorff topological space. Let $C \subset X$ be a compact subset. Prove that C is closed. Is the conclusion true even if X is not Hausdorff?

B5. Let $A = S_1 \cup S_2$, where $S_1 = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}$. Consider the set $B = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2, -2 \leq x \leq 2\}$.

(a) Are A and B homeomorphic?

(b) Are A and S_1 homeomorphic?

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B6. Consider the sets $A = \{(x, y) \in \mathbf{R}^2 \mid 0 \le x \le 1, y = 0\}$ and $B = \{(x, y) \in \mathbf{R}^2 \mid 2 < x < 3, y = 0\}$. Find a surjective map from $\mathbf{R}^2 \setminus A$ to $\mathbf{R}^2 \setminus B$.

B7. Suppose four cards are drawn one-by-one from a standard deck of 52 cards. What is the probability that the third card is either a King or a Queen?

B8. Let μ be the Lebesgue measure on the interval $X = [2, 3]$. For $x \in X$, define

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{a}{2^n}, \quad a, n \in \mathbf{Z}; \\ x^2 (\log_e x)^2 & \text{otherwise.} \end{cases}$$

(a) Is f Lebesgue measurable?

(b) Is f Lebesgue integrable?

(c) Is f Riemann integrable?

B9. *(a)* Prove that $\mathbf{Z}$ and $\mathbf{Q}$ are not isomorphic as abelian groups.

(b) Give two different reasons to show that $\mathbf{Z}/6\mathbf{Z}$ is not isomorphic to the permutation group S_3 of the set $\{1, 2, 3\}$.

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B10. Let S_3 denote the permutation group of $\{1, 2, 3\}$. To each $\sigma \in S_3$, associate a map $L_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L_\sigma(x_1, x_2, x_3) := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.

(a) Show that L_σ is a linear operator on $\mathbb{R}^3$.

(b) Find a common eigenvalue and a common eigenvector of all the σ in S_3.

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ENTRANCE EXAMINATION, 2010

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Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

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PART—A

Answer **all** questions. Each question carries 6 marks

A1. Find the extension degree of F over $\mathbb{Q}$ when

(a) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{4})$

(b) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$

(c) $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6})$

A2. What are the points of intersection of the following two curves in $\mathbb{R}^2$?

$$y = \begin{cases} x \log_e x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(curve 1)}$$

$$y = x(1-x) \qquad \text{(curve 2)}$$

Compute the area enclosed between these two curves.

A3. Let

$$X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 2\} \quad \text{and} \quad Y = X \setminus \{(1, 0)\}$$

(a) Is X compact?

(b) Is X connected?

(c) Can one find a bijective map from X to Y?

A4. Evaluate

$$\int_0^\infty \frac{x \sin(ax)}{x^2 + b^2}\, dx$$

where $a, b \in \mathbf{R}$, $b \neq 0$.

A5. Let G be a cyclic group.

(a) Prove that any subgroup of G is cyclic.

(b) Suppose that the order of G is n and $d \geq 1$ is a divisor of n. Prove that G has a unique subgroup of order d.

(c) Suppose that ϕ is the Euler function defined as

$$\phi(m) = \#\{1 \leq k \leq m \mid \gcd(k, m) = 1\}$$

Prove that $\sum_{d \mid n} \phi(d) = n$

PART—B

Answer **all** questions. Each question carries 4 marks

B1. Construct a group G with the following two properties :

(a) G is uncountable.

(b) Every element of G has finite order.

B2. Let R be a ring with unity and let $x \in R$.

(a) If x has a left inverse y and a right inverse y', then prove that $y = y'$.

(b) Is it necessary that if x has a left inverse in R, then it is invertible?

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B3. Evaluate :

(a) $\lim_{x \to \infty} (x^{-1}e^{2x} - 1)$

(b) $\lim_{x \to \infty} x^{-1} \sin x$

B4. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that f has a fixed point, that is, show that there exists $x \in [0, 1]$ such that $f(x) = x$.

B5. Find the value of

$$\lim_{n \to \infty} \int_0^1 x^n \cos \frac{\pi x}{2} \, dx$$

B6. Let $\mu_1, \ldots, \mu_n$ be measures on a measurable space (X, Σ) and let $a_1, \ldots, a_n$ be non-negative real numbers.

(a) Show that for any $E \in \Sigma$, $\mu(E) = \sum_{i=1}^{n} a_i \mu_i(E)$ also defines a measure on (X, Σ).

(b) Find the conditions which ensure that μ is a probability measure.

B7. Let T be a bounded linear normal operator on a Hilbert space. Given that

(a) T is invertible

(b) $\|T\| \leq 1$

(c) $\|T^{-1}\| \leq 1$

show that T is unitary.

B8. Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$ and let $\mathrm{sp}(A)$ denote the spectrum of A.

(a) Prove that if $\lambda \in \mathrm{sp}(A)$, then $e^{\lambda} \in \mathrm{sp}(e^{A})$.

(b) Show that for a bounded linear invertible operator B on $\mathcal{H}$, $\mathrm{sp}(A) = \mathrm{sp}(BAB^{-1})$.

B9. Let (X, τ) be a topological space where $X = \{1, 2, 3, 4\}$ and

$$\tau = \{X, \emptyset, \{1, 2\}, \{2\}, \{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$$

(a) Prove that X is not Hausdorff.

(b) Find the smallest Hausdorff topology on X which contains τ.

B10. Is $\mathbb{R}^2 \setminus (\mathbb{Q} \times \mathbb{Q})$ path-connected?

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